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Reading Sociologically

Introduction

This is a journeyman piece. It makes no big claims for its proposals and tries to make as plain a case as it can for them. The objective is to outline an approach to the reading of sociological research reports which treats such reading as the sense assembly of plausible accounts of the phenomena they depict. The depictions given are sociological objects and the phenomena they represent are social ones. We will suggest the process of moving from the one to the other is a matter of data reduction, a process which thereby creates an abstraction gap. In assembling their sense or understanding of a sociological report, we propose readers should seek the extent to which the span of the abstraction gap reduces or increases the plausibility of the account being given. We will summarise our approach in a reasonably self-evident heuristic and illustrate it by reference to a few studies. The studies are drawn from the sub-domain of mathematical modelling in Sociology. There is a reason for this. Mathematical modelling in Sociology deliberately couches phenomena in terms of well-attested formalisms usually derived from the physical sciences. These are abstractions. Sociology's data have their origins in the concrete accounts of their experience of social phenomena which ordinary members of society provide as input to sociological research processes.[1] Making this stipulation means, for the moment at any rate, we can avoid having to

[1] This input can take many forms and be more or less distant from the experience described. Personal accounts, documents and reports, survey responses, participant observation, experimental set ups and the analysis of official statistics all require some form of coding before being 'mathematised' in an analytical model.

address issues of realism, naturalism, subjectivity, faithfulness to the phenomena and so on which would otherwise complicate our task. Mathematical models are just that: models. No-one wants to nor should confuse the model with that which is modelled. There is another advantage to be had. The source of our heuristic lies in philosophical thinking about the epistemology of Physics. Since it borrows so much else from it, we can make the reasonable assumption mathematical modelling in Sociology is committed to the same epistemological virtues as Physics.

Section 1. What is a Well Posed Problem?

Stephan Körner once observed Pure Mathematics has parted company with perception [Körner 1968]. We think he meant it had parted company with the Natural Attitude. To speak loosely for a moment, the world of pure mathematical objects (e.g., infinitely small points, converging parallel lines, denumerably infinite numbers, trans-finite numbers, multiple systems of numbers including countable numbers, negative numbers and a number for the absence of number, etc.) is not derived unmediated from the object world of daily life. The divergence, however, is of no matter until we seek to apply Pure Mathematics back onto the world of common sense. When deciding how much concrete will be needed to fill a post hole, work out how much carpet to order for a room or the time it will take to make a journey by car, we find ways of bringing the abstract objects and associated manipulative techniques of Pure Mathematics into alignment with the activities of daily life. We do so by relaxing rigour and definitional rigidity to allow approximation and relative goodness of fit. Körner suggests this involves substituting inexact for exact concepts.

As we will see, with Sociology at least, what is going on is a bit more complicated. Study after study has demonstrated the sociological attitude involves the use of sociological concepts which depend on elements of the Natural Attitude. In its actual practise, the sociological attitude is a comingling or confluence of technical sociological conceptualising and the Natural Attitude of common-sense life. Doing sociological investigations could well be described as naturalistic sociologising. Applied mathematical modelling in Sociology comprises two orders of transformation therefore: from common sense modes of understanding of the social into a naturalistic sociology's modes of understanding and from there into a mathematised naturalistic sociology's mode of understanding. These transformations are not incommensurable. Much is preserved, though some things are set aside or, as we will say, effaced. As we have said, our interest is in the character of these transformations and their implications for the sense assembled plausibility of the analyses they sustain.

Somewhere on the edge of the novice sociologist's learning strategy should sit two questions. The first is: What is it about the social that makes a 'sociology' possible?[2] In other words, what is it about the metaphysical characteristics of social phenomena which mean they are amenable to being corralled and subjected to systematic investigation? The second is: What are the ways in which the social are made investigable by the kinds of sociology which are generally practised? Although related, these two are not the same. The first is *philosophical* and its hinterlands are issues of perception, cognition, ontology and, of course, epistemology. The second is *socio-methodological*. It has as its hinterlands the socially organised character of the practices by which investigations of the social are carried out. Our starting assumption concerning issues of practise and metaphysics is that for most sociologists most of the time, solutions are culturally given within the standard forms of the discipline they adhere to. They come in conventional packages and are simply how 'sociologists like us do what we do'.

The focus here is on the institutionalised choices made regarding mathematical modelling in Sociology. We are agnostic with regard to the relative 'value' (on whatever dimension you want to construct it) of the examples we review. Rather, borrowing a term of Brian Smith's [Smith 1996], we are interested in them as particular modes or methods for "registering" the social which have been adopted within the discipline. Inevitably, such registrations are abstract and generalised. Our questions are about the constraints or requirements the use of these descriptions might impose on investigations. Clearly this does not mean we should demand all accounts be formed in the same way. They are, after all, different kinds of sociology and address different kinds of perfectly proper sociological objectives. But, since they are classes of generalised description, it is reasonable to ask how they stand as representations of the phenomena they encapsulate.

We will lump the tactics used to produce these renderings under the label "strategies of effacement", a notion we have taken from Mark Wilson [Wilson, 2017a; Wilson, M., 2017b]. Strategies of effacement are practices by which phenomena of interest are distilled and fixed so they can be subjected to investigative scrutiny. What we are looking at are the practices enabling the mathematical formulation of sociological problems evident in the instances of mathematical sociology we examine. The notion of 'practice' is a term of art in ethnomethodological accounts of professional work. We are not intending a fully formed ethnomethodological analysis. All we are doing is illustrating one way in which ethnomethodological concerns with sociological description

2 The term 'sociology' in this sentence does not designate an existing body of disciplinary practice nor some extant sub-domain. Rather, it means *any* sociology which might be made possible. The sociologies we have are just a (small) subset of the sociologies we might have.

as a mundane activity (its recognisably plausible reasoning) might be reflected back into the body of professional Sociology as a positive contribution to Sociology's own self-understanding.

A Position and Some Terminology

Guides to methods of mathematical analysis usually address problems by citing examples rather than their characterisation. To pursue the issues we have set out, we draw on some of Mathematical Physics' terms and concepts, albeit with the recognition that they will have to be adapted to be serviceable across the range of endeavours we have set our sights on. One particular resource is Jacques Hadamard's [Hadamard 1923] reflections on what constitutes a "well posed problem" in the mathematical analysis of physical systems. The domain he looked at was differential equation modelling and what is known as "Cauchy's problem". However, before we can turn to the details of analysis, we must consider the way the 'small world for analysis' is constituted. As already intimated, we call this the process of *registration*.

Each discipline has its own ways of constructing its processes for collecting data, transforming them into phenomena, and then presenting analytic results. All these ways construct a world-for-analysis. When turning to the social world as target for investigation, sociologists 'bracket' their normal, everyday understandings of how social life operates and construe the social processes they are interested in sociologically. The term "registration" is Brian Smith's [Smith 1996] and refers to the experience of directing one's attention toward objects rather than (passively) perceiving them.

> To register the world....is to do or be oriented towards the world *in such a way that it presents or arranges or constitutes itself as a world.* [Smith 1996 pp. 194-5 italics in original]

Smith goes on to say that registration not only refers to attending to individual objects and subjects but includes cultures, language communities and any other collectivity or social institution. What makes 'registration' particularly apt for our use are some of its connotations. Registration is an intentional act in the Brentano sense and hence someone whose registration is under discussion has a role in its selectivity. Registration does not just happen to you. Second, registration is not content free. Something is registered *as* a particular individual or type of object or process. Third, registration also carries the notion of alignment. To use a broad mathematical term, there is some mapping between the object and the mode of registration of the object. All these connotations testify to the premise Smith's use carries. The world-for-analysis is configured in large measure by the disciplinary and other cultural machinery we bring to it. It is important though not to see registration as purely or even primarily conceptual, not just the application of a schema or framework. As embodied beings, our awareness context is shaped both by how we think about things and by how comport ourselves towards them. *Mutatis mutandis*, this extends to sociological registration on those occasions where an important feature of a particular course of social action is the arrangement of or physical engagement with objects or other persons. Being clear about registration, allows us to move on to look at definitions of the domain and descriptions of objects in it. This we will call *characterisation*. Hadamard's analysis of Cauchy's problem provides a perfect guide here.

Cauchy's problem is this. Differential equations track change across two or more variables. When characterising a phenomenon in order to analyse it, one or more parameters of the distributions (what are called the 'beta' values) must be unknown. Empirically (under sets of specific real-world conditions), a particular set of values may provide a solution to the problem posed by the unknowns in the equations. However, our arriving at one solution does not mean we have exhausted the possibilities. In principle, an infinity of other solutions could be found along the curves described by the functions we have used simply because, by definition, these paths are infinitely divisible (the co-ordinates are given in the real number system $\mathbb{R}^N$). Cauchy's question was how to bound problems rigorously and precisely so that we can fix on a single solution. Clearly the *idea* that investigative problems should be well-posed is a general one. We confine ourselves here to what it might mean when applied mathematics is used to model in Sociology.

The essential feature of a well posed problem is to be found in the setting up of that problem (the 'form' the problem given for our analysis takes and the 'data' applied to it). This should be defined sufficiently clearly and precisely to allow a clean 'cut' or disengagement of our designated investigative objects and properties from the plenum of actualities encountered in the real-world situations from which we have drawn that problem. The set of arrangements (S) being examined must be sufficiently well extracted from surrounding environment (E) for S to be subjected to controlled manipulation and analysis.[3] Part of this cutting involves the "effacement" of detail. In discussions of their work, all researchers refer to this process as "simplification", "condensation", "idealisation", "abstraction" or in some similar way. These terms are perfectly

[3] Although this sounds like a version of the bench experiment, in fact it is the other way around. The bench experiment is just one way in which the well-regulated cut required for a well posed problem can be achieved.

serviceable but they lack the precision Hadamard demands. A well-posed problem demonstrates just what simplification, idealisation and abstraction *inter alia* should come to.

Hadamard suggests a well posed problem has the following characteristics.

1. A solution to the problem exists. That is, there is a known form or family of forms for representing the relationships depicted in the problem under a range of freely varied initial conditions.
2. The solution is unique. That is, the solution form does not lead to multiple possibly stable conditions (the potentially infinite solutions mentioned above) with no basis for making choices between them.
3. The solution responds in robust and stable ways to the range of data applied to it. The term Hadamard uses here is "continuously". By this he means there are no envisageable combinations of conditions among possible outcomes for the effaced system which generate 'chaotic' divergence thereby causing the problem to become "ill-posed".

For Hadamard, three further features or factors are required for a problem to be "well posed".

1. A set of boundary conditions marking the effaced transition of S from the plenum of E.

   Mark Wilson defines boundary conditions as:

   > Boundary conditions, roughly characterized, represent claims about how a certain portion of the universe interacts with its surroundings along their mutual boundary. [Wilson, M., 1990, p.566]

2. A set of initial conditions under which the effaced S begins to operate.
3. A well substantiated closed set of functions (for Hadamard these were always equations) describe the behaviour of S. By 'closed' he means there should be no gaps, elisions or uncontrolled collapses of phenomena in the account of S given under these functions. These closed sets are often referred to as "system laws".

Finally, general guidance is offered for the composition of the elements.

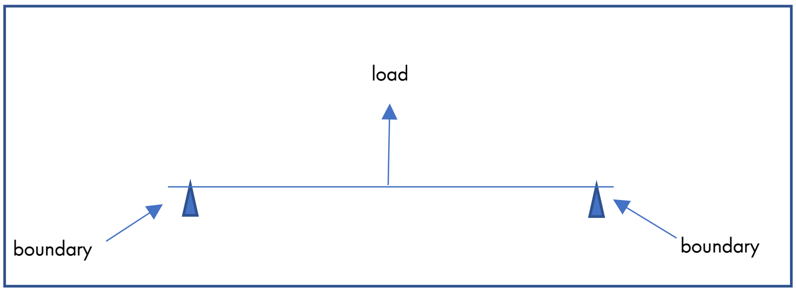
1. The set of boundary and initial conditions (**W**) should be sufficiently restricted to allow at least one possible solution to exist.
2. The boundary and initial conditions should not be so restricted that multiple possible solutions exist.
3. The form of the function used and the nature of the conditions selected should be matched otherwise the requirement of continuity may be undermined.

At first sight, it might seem Hadamard is trying to reduce applied mathematics to recipe following and rules of thumb, but this is far from the case. As he makes clear in his book *The Psychology of Invention* [Hadamard 1954], what we have just described is an orderly framework within which mathematical imagination, inspiration and perspiration can be channelled. Unfortunately, his views on the sources of this imagination and inspiration are very much of their time. Such reservations, though, should not undermine the value of his description of the formal conditions for 'well posed problems'.

A Toy Example

Suppose we are interested in the behaviour of a harp string. The string is connected at each end to fixed pegs and we want to describe how it vibrates when we pluck or tap it. This is a standard introductory example in applied mathematical modelling and, indeed, one Hadamard himself uses.[4] The system we are describing can be depicted like this.

Figure 1



The tapping or plucking of the string exerts a load causing it to deform. Once the load is released, over time the string returns to its original position.

What have we effaced in this model? The short answer is everything except the length of the string, the position of the pegs and the force of the loading. Other factors (well known to affect

[4] The version we are using is Wilson's. It is a favourite of his and is used in several places. The most technical is [Wilson 2006] whilst a more general recent introduction is [Wilson 2022]

real strings on real instruments) such as air pressure, humidity, the type of material of the frame and the relative condition and composition of the string have been severed off. We'll come back to the accommodation of these factors later.

What are the boundary conditions laid down for the problem? They are, first, that the string be fixed in position. Plucking or tapping it does not cause the pegs to move or the tension to be reduced. The system stays stable. This is known as a "Dirichlet condition" since it determines the values of the system at its boundary. At the pegs, movement of the string is zero. We do not have to try to work out just how much the pegs resonate as the travelling wave of restorative force hits them once the load is removed nor what that instantaneous deceleration does to the shape of the travelling wave. The second condition is the set of the precise co-ordinates of the load's position and the load's magnitude. Where the load is placed sets the length of the string each side of the pluck point and thus the pattern of wave interference as the waves meet each other on their return from the pegs. Third, the movement of the string in response to the pluck or the tap is vertical to its original position. In other words, the string does not warp. Such warping would require an analysis of the dynamics of the forces along the interior of the string. These two conditions are also Dirichlet with the latter allowing what Wilson calls "physics avoidance" of tricky, perhaps unmanageable, conditions. The clearly stipulated features of our well posed problem leave us with a set of conditions (**W**) which are precisely defined and bounded.

What, then, are our initial conditions? They are (a) the position and length of the string (its coordinates); (b) the position and magnitude of the pluck or tap; and (c) its precise timing. Obviously, what we are tracking are the dynamics of the string's movement over time (in physics-speak, the trajectory of its states from t_0 to t_n).

Finally, what system laws can we use to track the string's trajectory? These are the standard wave equation, Newton's basic F = ma 'law' and Hooke's 'law of restoration'; all three being used together under the stipulation of no warping. The wave equation has the following general form, the components of which can be filled out by plugging in measurements processed by the above laws.

$$\frac{dy^2}{dt^2} = \frac{c^2 dy^2}{dx^2}$$

In providing for the curvature of the string and the restorative force of the loading, the combination of system laws achieves the required equational closure. We can use it to project forward the states of the system over the relevant time frame in which the boundary conditions remain in place.

This gives us what we can call "descriptive sufficiency" for the problem we have posed over the time frame we have defined. The description is sufficient to allow an appropriate solution to our well posed problem.

What happens if we want to alter the boundary conditions, say by introducing a violin-like 'bridge' over which the string passes? This introduces a new set of states (**P**) into the model. The challenge is to define a new set of Dirichlet conditions which ensure the integrity of the system is maintained under **W** + **P**. This will require any descriptions we give of **P** to be consistent with our descriptions of **W**. In our toy case, we will require further characterisations in the terms given by the system equations. The demand for the maintenance of structural integrity in **S** means we cannot simply make our effacing cut anywhere. The choice we make must respect the demand for consistency.

In all of this, as Hadamard and practising bench scientists are well aware, the nature of the measured data is critical since the well posed character of the problem depends upon continuity with that data. It is here that the tricky topic of measurement error enters the discussion. Even small errors in measurement can lead to large divergences as differentiation unfolds. To combat this outcome "regularisation" adjustments are introduced. We will not discuss such technicalities here.

Finally, although there are constraints on the placing of the cut, it is important to note the modalities of effacement (what is severed off and what is retained) are interest driven. The process is not a boiling down to essences but a selection of focus. Given the starting point is the plenum of the world, how those interests are defined is open—though as we have already suggested, in large measure the process is usually institutionally or culturally given.

So, what have we got here? We have a general strategy for selecting the constituents of a "small possible world", the system S, to which we can address our interests. This selection consists of an orderly filtering or effacing of phenomena in which we are not interested or cannot handle and then the precise definition of the boundary conditions under which the features of interest can be presented. In dynamic systems, the initial conditions for analysis are set as well. Finally, we must have a sufficiently complete analytic apparatus (what we called "system laws") to cover any possible state the system might attain given the boundary and initial conditions set and the functional form of the descriptive apparatus.

The issues in constructing well posed problems are about walling off complexities we either don't want or can't manage. These complexities may be scope-related (the entanglements of the environment such as those we mentioned) or they may be scale-related (the effects of the vibration along the whole length of the string rather than the small-scale effects of molecule realignment in the cross sections of the string at points of warping).

The Heuristic

From the above discussion, we can extract the following key components of a well posed problem.

1 Problem Statement.
   a. Registration and characterisation of phenomena.
   b. Notation or descriptive formulation.
2 Problem Specification.
   a. System laws and relevant generalisations assumed to be in operation.
   b. Boundary conditions fixing the scope of 'the world under investigation'.
   c. Strategies of effacement filtering out irrelevant or irresolvable issues.
3 Analytical Protocols.
   a. Analytic procedures applied to effaced descriptions.
   b. Analytically derived results interpreting over the results of those procedures.

Although this might seem like an orderly framework for a coherent narrative concerning a problem and its solution, in practice the components are often assembled in different sequences. However, using this check list as a guide, a careful reader can reconstruct the contents of a sociological report's propositions, descriptions and summaries to sense assemble the sociological significance (both meaning and import) of the material which is presented. This will be not just be in terms of the integration and coherence of the piece but also its relative alignment with other similar research reports whose contents may have also been sense assembled in this way. In following this approach, a newcomer would soon have a good grounding in how to reason sociologically and, in time, what the state of the field in any domain might be. If adopted broadly, it might even encourage more effective calibration of sociological reports and thus be a practical contribution towards the discipline's achieving a degree of probativeness. Something it does not currently possess.

Section 2. Some Illustrations of the Heuristic

We will now take our heuristic and see how it fares when applied to a small selection of ways social phenomena are registered and analysed under various forms of mathematical sociology. The cases we look at are taken from some of the major domains of mathematical modelling in Sociology. We will present the cases in relatively brief vignettes focusing only on how they are worked through. We will not provide their disciplinary background, consideration of their strengths and weaknesses as solutions to the problems taken up nor review subsequent developments in the relevant fields.

ABM and Racial Segregation

Our first example is probably the most abstract and certainly the most controversial. It is Thomas Schelling's description (what these days is called an 'agent-based model') of the unintended generation of residential segregation by race resulting from members of a community acting on their preferences for whom they would like as neighbours. Whilst the 1969 paper [Schelling 1969] is perhaps the more well known, the developed account of his simulation is to be found in the 1971 version [Schelling 1971]. In both papers, Schelling presents his results as three scenarios. Using the terminology we have adopted, we will say these scenarios represent related sets of registered and characterised small worlds and their boundary conditions, though the third is really a different problem. In each case, the scenario shows how segregated patterns of residential distribution emerge given a range of initial conditions applied to given boundary conditions. Central to the array of scenarios is the nature of the space within which the distributions are considered. The argument Schelling presents moves from 1 dimensional, to bounded 2-dimensional and unbounded 2-dimensional spaces.[5] The simulation is adjusted as we move through the types of space.

We start by scoping how Schelling registers the small world. First, he posits a formal analogy between the social processes of racial segregation and the emergence of unanticipated and hence unintended structures (so-called "hidden hand" consequences) in market systems when buyers have well defined and ordered preference for goods and services together with free choice over how to exercise those preferences. When it emerges, this kind of segregation is "unorganised". By calling the analogy 'formal', we mean the same abstraction can be used to

[5] Just to be clear, "space" in this and related discussions does not necessarily refer to our ordinary 3-dimensional sense of space. Instead, it refers simply to a set of objects and the defined relationships among them.

represent the general principles at work. In choosing racial segregation as the social phenomenon for his model, Schelling defines the second of his boundary conditions. The criteria for segregation are said to be binary (white and non-white (black)), exhaustive (everyone is either white or non-white), and immediately recognisable (expressed as skin colour). In his first two scenarios, Schelling adds an important third initial condition. No-one in the community has a preference for the overall mix in the neighbourhood even though they do have preference for the mix of their own nearest neighbours.

ONE DIMENSIONAL BOUNDED SPACE

The following further boundary conditions are laid down for this scenario.

1. The space or neighbourhood is linear and can be continuously (i.e., infinitely) partitioned (operates like the number system). With appropriate expansion, space can always be found between two contiguous areas.
2. Each member of the neighbourhood has a residency preference or tolerance for the proportion of the category opposite to themselves living in their immediate environment. If that preference is met, they will remain in their current position. If it is not, they will move within the community to where that preference does hold.
3. Movement is frictionless and unconstrained (in other words, costless and always possible because of the infinite divisibility of the space).
4. The number of residents in the neighbourhood is fixed.
5. Each triggered movement is to the nearest 'space' which satisfies the member's tolerance level. This is achieved by assuming movement is facilitated by everyone 'budging along' to allow the incomer to be inserted and the space the incomer left thereby to be filled.

In this scenario, we have a fixed number of members of binary categories arrayed linearly with 'motivating' conditions under which they will stay in place or move. Schelling now imposes two "system laws".

1. Given the binary nature of the categorisation, both categories cannot be in the majority.
2. The tolerance schedules are functions holding the ratios of the categories' preferences. Once an individual's neighbourhood reaches the upper threshold, that

individual's movement is triggered. Schelling left these functions defined informally. Haw and Horgan [2018] have recently formalised them as:

$$\frac{Y}{X} = XR_x\,(X) \quad [X's\,preference\,ratio]$$

$$\frac{X}{Y} = YR_y\,(Y) \quad [Y's\,preference\,ratio]$$

3. The dynamics of the process can, therefore, be expressed as a pair of differential equations.

$$\frac{dX}{dt} = X[XR_x(X) - Y$$
$$\frac{dY}{dt} = Y(YR_x(Y) - X$$

It is important to note the schedules comprise two different types of function. One is the paired preference orders just described. The other is an *if...then...* decision rule. It is the decision rule which triggers the action if the threshold is passed. The conjunction of the two functions drives the simulation.

There are two Dirichlet conditions. The first is the definition of the actor/agent/member of the community as a finite state automaton with 4 (paired) states: satisfied/unsatisfied; move/don't move. This is the severing from the ordinary conception of a social actor acting in a social environment. The other is the stipulation that the automaton's "psychology" be limited to just one component, the preference rule. This is a necessary scaling effacement and "avoids" the complexities of having to derive a "sociology of preference coordination" which would otherwise be required. It is equivalent to our setting aside the 'interior' of the string in our toy example.

The initial conditions Schelling uses are:

1. The ratio of the categories is 1:1.
2. The placing of the categories along the space is random.
3. The preference ratio is 50:50 for a local neighbourhood of 8 nearest neighbours. For any member of the neighbourhood to be satisfied the run length of nearest neighbours in their own category must be 5 out of 9.

Having posed the problem, Schelling iterates over the distributions shifting the unsatisfied members of the community according to the rules. Each set of moves constitutes a new configuration of the racial composition of the neighbourhood and hence a new state. Over the iterations, the nearest neighbour run length proportions drive the degree of racial clustering of the community, that is, the extent of unintended racial segregation.

Three things are worth noting at this point.

1. Even this highly simplified model has far more elaborate boundary and initial conditions than our 'toy' model. This may speak to the difficulty of easily severing off parts of the social world.
2. The general structure of the model is very little different to Hotelling's famous model of the market for retail space in town centres in terms of the operation of supply and demand in unrestricted 'free' markets such as stock exchanges [Hotelling 1929]. Although the natures of the driving functions are different, segregation, the distribution of retail space and stock markets all rely on preference orders.
3. Apart from the tautology about majorities, the algorithmic system laws are nowhere near as empirically well secured as those of those covering the toy model. The best one could say of them is that they are convenient fictions constructed for the simulation.

Two-Dimensional Bounded Space

For this scenario, Schelling redefines the boundary conditions and thereby significantly shifts the cut. The space becomes delimited and 2-dimensional. Schelling uses a 13 x 16 checkerboard as a working definition of the space he is imagining. As we saw in our initial discussion of well posed problems, any relaxation of the austerity of the definitional severing raises questions concerning the maintenance of the integrity of the system under view. Adding a second dimension violates the boundary conditions of the first scenario simply because it is no longer a 1-dimensional space. Consequently, Schelling has to adjust the boundary conditions as follows:

1. The working concept of space changes from a relative one (who is next to whom) to an absolute one (everyone sits on a space defined by bounded integer co-ordinates (1:13, 1:16)).

2. A fixed number of vacant spaces is always available for occupation. When a member moves to occupy such a space, a vacancy occurs in the space left behind.
3. Movement is limited to the nearest vertical or horizontal vacant space.

Under these set-up conditions, the system laws remain the same.

The initial conditions proposed are:

1. The nearest neighbour ratio is 50:50 membership of the two categories.
2. The run length of nearest neighbours is defined by the 8 surrounding neighbours.
3. The initial distribution of members of the categories is random.
4. An ordering of moves and what counts as their 'periodicity' is defined. For example: a turn might consist in making all the moves for one category of dissatisfied members whilst the next turn moves dissatisfied members the other category. Alternatively, moves might be made in a top-down, left-right or similar fashion. Or again, the ordering could be random.

As with the first scenario, in different runs of the simulation Schelling varies initial conditions such as the tolerance ratios and whether they are balanced, the numbers of members in the community and the mix of the categories. Each run produces a different unintended configurational state or zonation resulting from the shifting positions of members of the population as they scoot around the board. The properties of the zonation define the extent of the segregation created.

Clearly, this scenario is an extension of the first and its dynamics are very similar. For both scenarios, the ultimate stable state is a racially segregated neighbourhood.

Two-Dimensional Unbounded Space

At this point, Schelling changes the problem entirely. Instead of patterns of zonation, he focuses on the structure of the population under conditions of competition for inclusion. This makes the tolerance schedules even more like the utility functions of economic theory (willingness to tolerate being the analogy of willingness to pay) and the allocation process even more like a market for residence. What Schelling is interested in are the dynamics of demand for the products in such a market; that, is, the final category distributions within 'black' and 'white' residential areas.

The new boundary conditions are:

1. The space (i.e., neighbourhood) is wholly abstract and so the spatial constraints can be entirely dropped. As many as are willing to live there, can live there. The property of "nextness" in the 1- and 2-dimensional sense is an irrelevance as is the question of absolute numbers. Now all that is at issue is the mix or ratios.
2. Both categories are defined by frequency distributions for tolerance. These distributions may not be the same for both categories. In other words, preference orders vary across the members of each category (even if only marginally). There is no upper bound on the ratio of their own category they will tolerate but there is an upper bound on the ratio for the opposing category. That upper bound will be a position along the tolerance function (willingness to tolerate). When someone perceives their neighbourhood has reached that point, they move. It follows that while members of each category could well wish to live in neighbourhoods made up of different ratios including one exclusive to their own category, no member of either category will remain when they are surrounded only by the other category. Incremental shifts in the ratios generate movements in and out of mixed category clusters as individuals perceive the neighbourhood to be more or less attractive. At some point at less than a 99% mix there are no members of a category willing to tolerate the mix of the other (being the only black among whites or vice versa).
3. Everyone knows the precise ratio of the categories in the neighbourhood, but they do not know the tolerance functions of other individuals. This means there can be no 'futures market' for residential access. This is the analogue of the idealisation of perfect information in a free market.

The first and the third of these conditions are clearly effacements. The first sets aside issues of space constraints, transfer frictions and so on. The third sets aside uncertainty and hence the related (social) psychological attributes which otherwise would have to be modelled. In other words, the finite automata have been re-specified.

Under these new boundary conditions, the initial conditions required are more limited. They are:

1. For each run of the simulation, the ratio of the categories for any individual is a re-defined constant (k).
2. The tolerance functions are linear and the upper bound are defined.

Because of the initial conditions, the system laws are amended slightly. Instead of

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{R}_x(\boldsymbol{X})$$
$$\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{R}_y(\boldsymbol{Y})$$

we have

$$\boldsymbol{Y} = a(\boldsymbol{X}(1 - k\boldsymbol{X}))$$
$$\boldsymbol{X} = b((1 - k\boldsymbol{Y}))$$

Plugging defined ratios into these equations produces parabolas which act as a 'pay-off' functions or a 'return on tolerance' curves. Tracing each function on a two-dimensional space produces distributions such as the following:

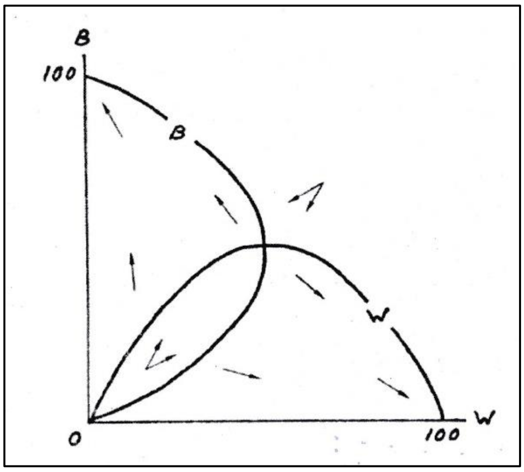


Figure 2 From Schelling (1962, p.492)

The intersection of the parabolas is a zone of 'static viability' whilst outside the intersection are zones of 'dynamic movement'. Note there are points on the mapping where the values are 0:100 and 100:0 indicating states of total segregation. As the parameters of the initial conditions are altered, the shapes of and interactions between the parabolas change. In this scenario, Schelling's analysis is a working through of alterations to such parameters as limits on the size of each population, variations in the tolerance functions, limitation of the ratios of the categories (essentially rationing membership for either category) together with possible perturbations. Each run of the model produces a different mapping of the zone of intersection and hence different patterns of segregation.

Discussion

The absence of direct empirical reference in Schelling's model means we would be justified in describing it as an exercise in pure mathematical sociologising rather than an applied mathematical modelling. It uses social or sociological labels for mathematical objects (e.g., racial mix preferences as functions). Because it is such an exercise, it is relatively straightforward to re-cast the analyses within the well-posed problem framework. The mathematics used is not derived from the sociology but self-standing and the sociology is plugged in to it. The processes of effacement in the definition of boundary conditions and the specification of initial conditions shape the social phenomena so that the mathematics can be applied to them. Of course, this is equally true of our own toy example but with two significant differences. First, the mathematics applied in the toy case was developed to solve problems of the kind to which it was applied. Second, the processes of effacement were 'structure preserving'. What was carried over in the severing were the defining conditions of the vibrating string as experienced pre-analytically. Schelling, on the other hand, re-characterises the pre-analytical problem with each scenario. This is not a point about the 'realism' of his analyses or their 'goodness of fit' to the experience of residential segregation but about the requirement to achieve a degree of conformity to a mathematical framework. It seems that ensuring the well posed character of the mathematical formulation of a sociological problem might come at the possible cost of weakening the preservation of structure within the sociological construal of the social. This is an issue to be held in mind as we turn to further examples.

SEMs and Career Patterns

The universal availability of high-powered personal computers and their statistical programming environments has made the mechanics of structural equation modelling (SEM) trivial—some would say too trivial. Here, for instance, is one knowledgeable observer.

> Modern SEM software makes it possible to crank out results for complex, networked models that are based on only the vaguest of intuition, and then to test these same intuitions with data that does not directly measure them—and to do it all without understanding the statistics. In the wrong hands, this is surely a recipe for bad science. [Westland, 2015, p.5]

We are not going to rummage around in all the issues Westland raises, but concentrate on just one class, those of problem set-up.

SEMs are systems of simultaneous equations describing the relationships between underlying and hence not directly observable (latent) structures. The relationships used are measured covariances among observed variables. They are often accompanied by illustrative, so-called 'path' or 'network', diagrams which order the relationships described. There are numerous flavours of SEM but a core two step technique is common. Measures of covariance are computed across a range of observed 'indicator variables' assumed to bear on a target problem. Principal Components Analysis (PCA) is then applied to that covariance matrix to derive factors or latent variables. From the latent variables so derived, coefficients for the associations are extracted and used to define a model for the (causal) 'structure' of the effects the latent variables have on each other. At their simplest, SEMs contain one more complicating step in their effacement strategy than ABM simulations. The selected indicator variables are severed from the entanglements of real-world experience and given mathematical form. That is step one. They are then subjected to processes of variable reduction and transformation as they are shaped into the model. That is step two.

Although the terminology varies among the different techniques, a common set of concepts are used:

1. A distinction is drawn between an 'inner' structural set of modelled relationships among the latent variables and an 'outer' theorised collection of indicator or measurement variables.
2. The inner/outer structure sustains a distinction between the exogenous and endogenous variables which comprise the total system of relationships. Exogenous variables are assumed (or known to be) statistically independent of the endogenous ones.

3. A decision procedure identifies factors either based *a priori* on what can be called "informed subjectivity" (termed 'reflective latent variables') or from the measures of association across the set ('formative latent variables'). As Westland implies in the quotation above, inexperienced use of reflective variables is a recipe for ill-posed problems.

The distinctions point to what is known in SEMs as the "identification problem". How that problem is resolved is the topic we focus on. Other interesting questions abound in the entrails of the machinery of modelling. We will not address them.

Here is the standard SEM diagram.

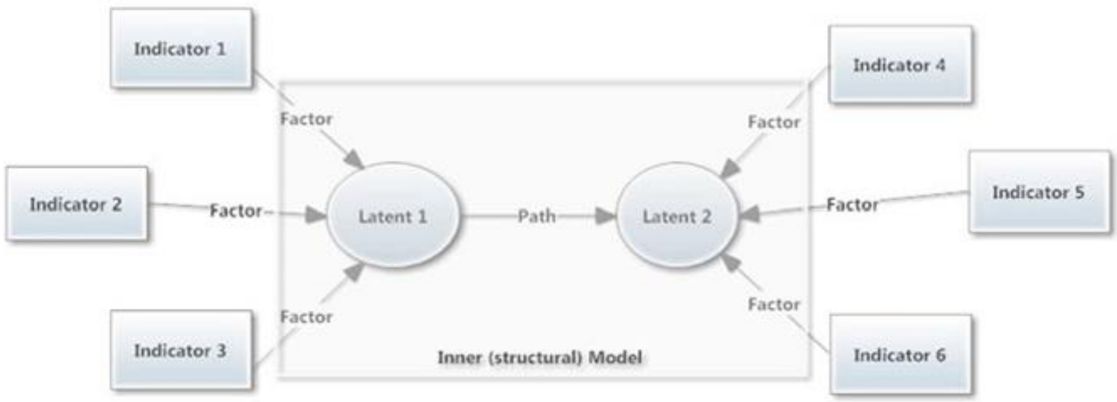


Figure 3 From Westland p.11

The indicators are the measured variables. The latent variables are the outputs from the PCA. The paths in the model are given by this notation:

$$Lat1 = f(Ind1 + Ind2 + Ind3)$$
$$Lat2 = f(Lat1 + Ind4 + Ind5 + Ind6)$$

The diagram brings out the first boundary condition very well. The relationships are acyclic. There are no feedback paths among the latent variables nor from them to the indicators. We can understand why this must be so by looking at what happens if we violate the condition.

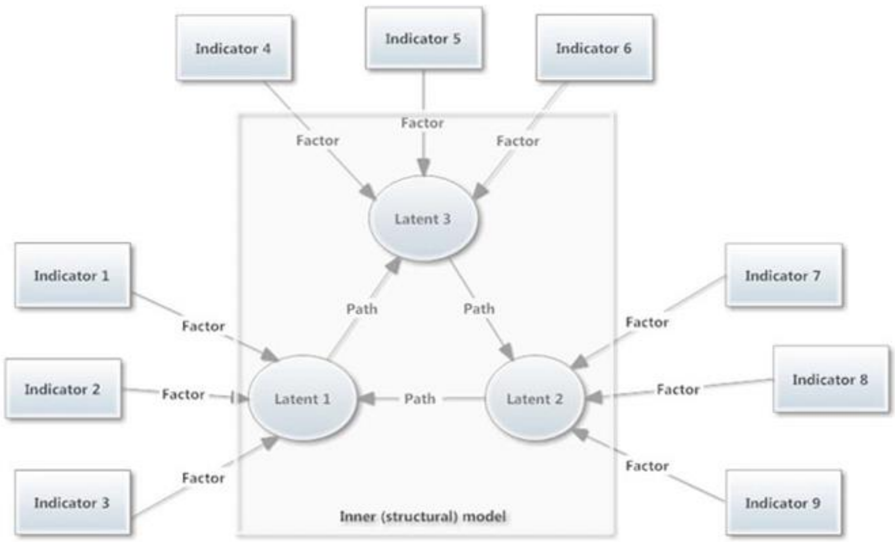


Figure 4 (Westland p 13)

Labelling the latent variables L1.....L3 and following the paths, we see L2 is influenced by a path from L1 via L3. In turn L2 influences L1. Thus L1 = *f*(L2+ L3 + I1 +I2 + I3). We can solve for the indicators because they have a non-recursive relationship to the latents. However, the relationships among the latents create what is known in computing as a 'race condition'. The initial value of the system can only be set by solving the system of latent variables for L1 but to do that we have first to compute L2 and L3. We can only break the circle by stipulating an initial value or using some other 'regularisation' technique. Alternatively, we can re-construct the model (in other words, start again).

We can see the second boundary condition by looking at the equations. Suppose we reduce the system (replacing L with the more familiar X and I with Z to ensure clarity). We end up with familiar simultaneous equations.

$$X_1 = b_1 Z_1 + u_1$$
$$X_2 = b_2 Z_2 + b_1 Z_1 + u_2$$
$$X_3 = b_3 Z_3 + b_2 Z_2 + b_1 Z_1 + u_3$$

The $u_{1\ldots3}$ are the residual or 'error' terms and refer to any unknown variance left after the variables are defined. If any of this variance is generated by one of indicator variables, then the exogenous/endogenous distinction is vitiated because the error term will not be statistically independent of that exogenous variable. This means the relevant indicator will have to be added to the endogenous set of variables. It is also usually assumed that the residual values are uncorrelated with each other (i.e., random). They are i.i.d: independent and identically distributed. The different forms such distributions can take is one of the ways various SEM techniques can be distinguished. The i.i.d condition is Dirichlet since it fixes the (lack of) relation between some parameters within the model (the latents) and the influence from the external environment transmitted through the indicators. Notice this is the only formal condition on the selection of indicators. Finally, and this is similar to the issue of race conditions, we must be able to construct the system of reduced equations so that at least one of them is distinguishable from the others (that is, not composed of exactly the same variables as the others).

To illustrate the setting of the initial conditions, we'll use one of Westland's examples. The data is invented but that has the advantage of facilitating clarity while not being part of its strategy of effacement. Suppose we are interested in the extent to which a person's health influences their success in a professional career. We might hypothesise a model like the one below where we have a set of health factors and a number of factors which influence career directly.

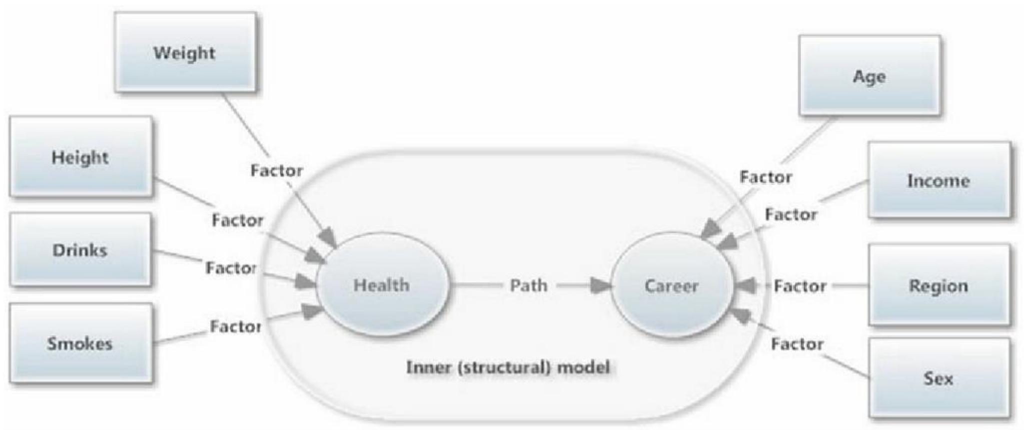


Figure 5 Westland p.68

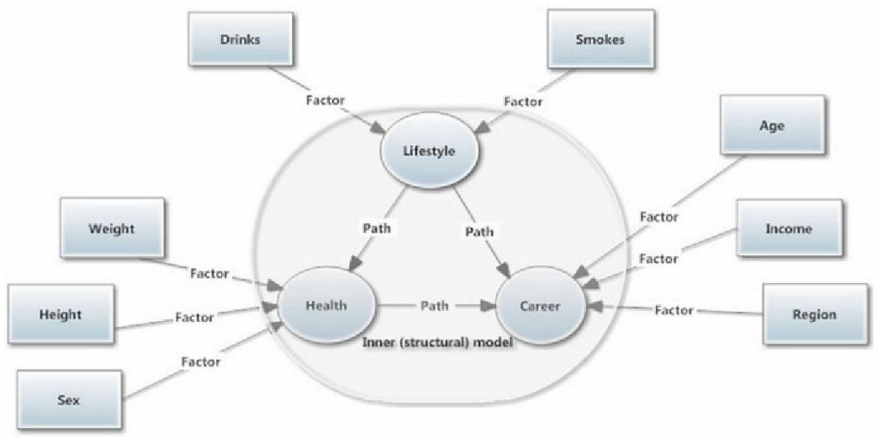
We operationalise our factors into measurable variables (sex being measured as male = 1 and female = 0) and survey a range of career professionals. At first sight, we might think the initial conditions ought to be the measures on the variables obtained by the investigative technique. In fact, the initial conditions for the model are the transformations of these measures constructed by computing pairwise variances across the indicators and deriving factor loadings for them. These factors are reduced variable vectors in the space defined by the indicators and are the principal components or latent variables. Each component reduces the variances of the pairs of measures to a set of single values (a vector of eigenvalues or eigenvector) which are the factor scores.

Thus far the transformation is 'mechanical'. PCA will produce as many eigenvectors as there are variables. So, how do we decide how many we need? The factor cut off is usually placed where the component variances of the vectors sum to 1.0 (known as the 'Kaiser criterion'). This is an informal guideline. Having made the cut, we examine our filtered components to see if they show clusters and/or whether any such high order cluster can stand for our hypothesised variables. Our analysis now depends on whether we have sufficient imagination or insight to discern and name an appropriate cluster to act as a latent variable or alternatively have the good luck to find indicator variables closely associated with hypothesised latent variables. We are looking for relatively higher order factor values (say a loading of 0.10 and above). In Westland's example, the PCA extracts 8 eigenvectors or principal components (L1.......L8) with the majority of the overall variance being carried by just 3. Looking at the composite indicators for the latent variables, we label the three 'Health' and 'Career' as before and the third 'Life Style'. Fixing of the initial conditions, then, is accomplished by mechanical computation, analytic 'nous' and culturally given norms (or rules of thumb).

The system laws are, of course, the simultaneous equations. Their form, but not their content, is given in advance. Each equation is a linear function (f(y) = a +bx). We have as many of them in the system as there are retained latent variables. The latent variables for our model are:

Life Style	-0.707drinks + 0.707sex
Career	0.327age – 0.6income + 0.729region
Health	0.436sex +0.722weight -0.538height

We can now reconstruct our model using the latent variables we started with and adding the third (discovered or 'formative') one.



We have two paths in the system: life style via health to career and life style directly to career. We can write the causal flow as

$$career \sim health + life\ style\ \ [career\ given\ \ life\ style\ and\ health]$$
$$career \sim life\ style\ [career\ given\ life\ style]$$

and calculate the relevant path coefficients (we'll ignore the mechanical technicalities of this step).

Path	Path Coefficient
Life Style ~ Health	-2.6
Health ~ Career	-4.92
Life Style ~ Career	-210.13

Discussion

Unlike our first example, SEMs are not a demonstration of specific theorems or pre-defined functions using sociological terms as designators. Rather, a statistical device is used as a computational machinery to transform observed data of social phenomena into a system of simultaneous equations whose functions are derived from the transformation. A standard format, the path diagram, is used to represent the results. In that sense, it is applied mathematical sociology. As a result, the analysis is far more responsive to the character of the observed data than the Schelling example. However, when we look at how the well-posed problem structure fits the set-up, we can see there is a fair amount of openness in the effacement process, especially in the selection of the initial conditions. Its plausibility structure rests on judgements about the fit of indicator variables to latent variables, assumptions about the i.i.d. status of error terms and informed judgements about the clustering of the factor loadings. These interpretations and judgements must be carried out skilfully and carefully if they are to bolster the plausibility of the 'causal narrative' concerning the relationships among the latent variables.

FQCA and Social Welfare

Qualitative Configurational Analysis (QCA) and its later variants break with mainstream quantitative social science. Rather than analysing variance across large numbers of instances of chosen variables (what are known as 'large Ns), it analyses qualitative and quantitative differences in those variables across a small range of cases. As its proponents put it, QCA offers a formal approach to the comparative analysis of small Ns. For its early developers, one of the key motivations was to find a way to deal with large scale social, political and economic phenomena for which replication under controlled conditions is morally, logically or practically impossible (either because history cannot be re-run, or because we can't use statistical control variables since individual observations are not i.i.d., or because controlled replication would violate social norms). Instead of solving differential and simultaneous equations, the formalism uses Boolean logic formatted as truth tables. As we will see, that description is something of a misnomer. If variable analysis is built around the arithmetical procedures of simultaneous and differential equations, QCA is built around the arithmetic of set theory. QCA's claims to being both quantitative and qualitative have made it attractive as a 'mixed method' spanning both approaches in Sociology. Because it focuses on and responds to the details of a small number of cases, it is often felt to be superior to the "blunt instrument" of variable analysis.

The general steps in analysis are easy to summarise.

1. Using your preferred theory, select a social phenomenon or process of interest (e.g., the conditions for stable democracy, the conditions for developed market economies, the conditions for sound financial decision making in Corporate Banks....).

2. Define the ideal type features of the phenomenon. Choose a title (like those just given) which allows flexibility in determining inclusion in the set of social objects defined by the ideal type.
3. Select instances/cases/ 'data points' which broadly satisfy the properties defined by the ideal type.
4. Define the 'scales' by which these properties will be 'assessed' and carry out the assessment for each case.
5. Translate these assessments into nominal categories (T: F or 0:1) for 'crisp' sets or scalars (in the range 0:1) for 'fuzzy' ones to produce a 'truth table' for the cases in terms of possession of the properties.
6. Configure the patterns of the assessments to produce an ordered list of the set members and set aside those which don't have a good fit to the configurations (the 'remainders').
7. From the surviving cases in the property set, extract the (sufficient and necessary) conditions for the ideal type processes defined above.
8. Use the initial theory to shape a narrative which accounts for the pattern of conditions and configurations for the array of cases in the set.

To bring out how a variant of QCA, *f*QCA (*fuzzy*QCA), can be reconstructed in terms of our heuristic for a well posed and analysed problem, we will walk through an example of the above protocol set out in [Ragin 2000].

THE DRIVERS OF WELFARE PROVISION

Various convergence theses suggest welfare provision (its universality and scope) should be increasingly standardised in the advanced capitalist democracies. The usual explanations rely on a claim about increasing homogenisation of political cultures generating two convergence forces: (a) marketisation associated with competitive pricing driving (b) low-cost provision forcing regression to minimally acceptable common standards. And yet the expected trends are not happening. The failure of the convergence is usually explained by traditionally entrenched 'leftist' political cultures, nationally strong unions and highly articulated democratic structures. All of which militate against single party dominance. Multi-party democracies are typically defined by a policy 'battle for the middle ground' where the provision of social welfare has had important electoral traction.

Ragin registers his world-for-analysis by listing examples of advanced capitalist democratic countries (mostly the ones you might expect). His question is about the varying combinations of conditions associated with the range of welfare provision on view. He identifies the following conditions: degree of generosity in welfare provision, existence of strong left parties, existence of strong unions, presence of a corporatist industrial system and sociocultural homogeneity. Advanced capitalist democratic systems with the conditions he lists define his sociological world. The proposition being tested is that the last four conditions produce, or at least shape, the provision of social welfare.

The definition of the conditions and the specification of their (logical) relationship locates where the analytic cut is being placed and the basis on which it is being made. However, the range of countries included is clearly narrow. Essentially, it is confined to states sharing North American and European ('Western') political cultures. Given we are talking about research carried out in the late 1990s, it might not be surprising China doesn't feature nor, perhaps, Russia. But why are South Korea , South Africa and Mexico excluded? Since one of the arguments being tested concerns the degree of sophistication of democratic structures, it could be argued these three should be included alongside the likes of New Zealand, Ireland and Norway to enable comparison on that particular variable. Ragin acknowledges the problem (p.296) by admitting the selection of countries is based on a stipulation—they must have been continuously democratic since 1945. Poland, Latvia, Lithuania and Estonia for example are thereby ruled out. East Germany is included as it is now part of a reunified Germany. It turns out, then, that the equivalence class of cases is not defined simply by the character of their political economy but also by the length of time this political economy has been in place. Although the restricted nature of the choice is acknowledged, its consequences are not explored.[6]

There is a second issue related to the (analytically necessary) severing of entanglements with the complexities of the real world. Most of the data used in the analysis consist either of unprocessed official statistics or are direct derivations from official statistics. Since one of QCA's central tenets is an acceptance that investigators have 'interests' which 'inform' the data they collect and the way they present them (this is part of their claim to being more 'realistic' than variable analysis), presumably Ragin would also accept state administrations have 'interests' which shape the data they collect and the way they present them. By relying so heavily on official statistics, entangled interests of the kind just mentioned are passed through into the analysis in a (statistically) uncontrolled way. Remembering Hadamard's concerns about the character of the

[6] For a discussion of reliance on overly homogenous equivalence classes, see [Mahoney 2022].

data fed into the system laws, such pass-through must have implications for how well posed the problem might be.

We now turn to the setting up of the boundary conditions. This is done by allocating the cases to ordinal positions on the properties of the ideal type. We'll look first at the set of countries with "strong left parties". As in all QCA listings, it is assumed that in assembling the set of cases with strong left parties, we are also assembling a list of cases which *do not* have strong left parties. In this example, we are not dealing with crisp sets defined by binary membership (yes, no, T: F). It is explicitly being assumed there is a range of 'relative inclusion' and 'relative exclusion' within the two categories. A country which is very definitely a member of one category will, reciprocally, be very definitely *not* a member of the other. Others will be more ambiguous. Generally, *f*QCA defines the set value of 0.5 as the 'crossover point' between being 'more in than out' and 'more out than in'. This median value is the universal peg for category membership. In other sets of course, should they wish to investigators could nominate different 'peg' values indicating where thresholds for full membership and marginality will be located. Such decisions reflect the investigator's judgement about the width of the value band for subset boundaries (for instance, 'almost fully in' and 'nearly fully out'). Set values, then, are ordinals defined as calibrations of set members in terms of what an idealisation for full membership of the property set defining the type would be. Although Ragin denies this is so, the investigator *chooses* where upper and lower bounds should be placed. Simply picking the highest or lowest GDP score or GINI index to determine set values for full inclusion (1) or exclusion (0) as measures of a country's wealth or poverty is not acceptable to fQCA. The consequent play in the ordination generates potential definitional problems, not the least being the difficulty of comparing different *f*QCA studies of what are ostensibly the same or closely related problems. In addition, it may well be impossible to trace the detail of the reasoning which drives the mappings between the input values derived from domain data and the set of membership values on which the sociological analysis rests.

The causal flow of Ragin's analysis has the standard structure. The independent variables (i.e., the explanans) are the distributions of the states according to the ideal type conditions. The dependent variable (the explanadum) is the value for generosity of their welfare provision. Like SEMs, the direction of flow is to the explanadum from the explanans. The necessary and sufficient conditions for variation in social welfare provision are derived from the distributions of the states across the sets. This means the 'assessment' of a state's values reflecting its positioning along the spectrum of 'generosity' in welfare provision is crucial and is the first place to look when discussing how the boundary and initial conditions are set.

Ragin uses two indices developed in Esping-Andersen's [1990] study of welfare and capitalism to define types of social welfare provision. These indices measure (a) the extent to which benefits are means-tested and (b) the separation of the benefits from labour market participation. Ragin transforms both indices into z-scores with the score for means-testing being inverted (means-testing being regarded as ungenerous). The resulting paired values on the indices are then averaged. Lower and upper thresholds are set (Ragin is coy about where, but by inspection we can infer it is at 0.1 and 0.9). With an average below 0.1, we can assume a case is 'fully out' and above 0.9 we can assume it is 'fully in' the 'generous welfare' type. This thresholding results in USA being 'out' of the set and Norway and Sweden being most definitely in. If the thresholds were set at 0.2 and 0.80 Canada and Australia would be drawn into the not generous set and Denmark into the 'definitely generous' set leaving Belgium being borderline to the latter.

The rationale for the z-score transformations is not clear. For statisticians, z-score transformation is a 'de-meaning' strategy which allows comparisons across very different distributions. The formula is

$$z = \frac{(\mu - x_i)}{\sigma}$$

where μ is the mean and σ the standard deviation. This results in a distribution in the range -1 to +1 around a mean of 0, so Ragin's transformation takes the averaged index values and re-arranges them in the range -1 to +1 with a mean of 0. Z-scores are dimensionless numbers (they do not measure physical, social or psychological 'variables') so what is supposed to be meant by 'averaging' them is not clear. Their value resides in the fact that because the variable they are derived from is assumed to be random, the scores can be treated as a probability density and so allow for comparison between distributions. A well-known statistical theorem states that for a Gaussian (normal) distribution, 68% of the distribution can be found within 1 standard deviation (z score of 1) from the mean and 95% with 1.96 standard deviations (z = 1.96) from the mean. Given this, we can compare the shapes (and hence the character) of very differently scaled distributions. Since QCA rejects "variable analysis", it is not clear why Ragin wants to treat welfare generosity as a probability density function. He could just as easily have scaled each case against the overall range of the variable. A scaling such as

$$(\frac{(max - x_i)}{max - min})$$

would have been much more obvious.[7] These scales would also be dimensionless, so the issue of averaging them remains.

If we now look at the calibration of socio-cultural homogeneity measured by indices of (a) religious and (b) ethnic and racial homogeneity, similar concerns arise. For each country, the proportion of the population by sub-group of religious affiliation and ethnic and racial self-identification is calculated. Each measure is squared (to avoid the total equalling 1 which would throw an error later in the calculation). The totals for each index are then summed and z-scored. Again, the z-scores are averaged. Setting aside the z-score oddity, we have a potential confounding issue here. In all the states listed, there is a strong collinearity between religious affiliation and ethnicity/race. Demonstrating these two are i.i.d would be quite a challenge, a fact which might lead us to think we are measuring the same 'latent' cultural variable twice.

So far, we have been raising issues of scoping, characterisation, severing and effacement with regard to just three of the properties or dimensions of Ragin analyses. It is clear the same concerns apply to the calibration of most of the others. One stands out not just for these issues but for the plausibility of the proposed evaluations. It is the condition of having strong unions. Three Nordic countries score highly (Sweden, Denmark and Finland). Italy, Germany and France are more marginal as is the UK. It is hard to understand how these latter four could be co-classified as a marginal subset. Union power in the UK has been in decline since the 1980s. In Germany, such power is institutionalised in socio-political and economic structures through arrangements such as Board membership—thereby making Germany much more like the Nordic countries. Italy and France have strong unions, to be sure, (they have recently been significant movers in the prevention of major state initiated social re-organisations of pensions and retirement age), but their power is exercised outside the political institutions. The term 'union power' seems to be too coarse to really be a useful indicator variable.

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[7] What would be lost would be the allocation of ordinals in the -1:0 range for the 'not S' class. But, as indicated above, that could be remedied (fixed up) by choosing a threshold somewhere in the low positive scalars (eg 0.25) to act as the 'peg'.

Table 10.6
Fuzzy Membership Scores for Analysis of Countries with "generous welfare states" (18 AIDCs)

Country	Generous Welfare States	Strong Left Parties	Strong Unions	Corporatist Industrial System	Sociocultural Homogeneity
Australia	.26	.25	.40	.17	.25
Austria	.72	.70	.64	.83	.67
Belgium	.79	.54	.84	.83	.29
Canada	.26	.00	.06	.05	.10
Denmark	.86	.85	.81	.83	.86
Finland	.76	.56	.86	.83	.72
France	.57	.12	.10	.33	.31
Germany	.68	.43	.20	.67	.30
Ireland	.67	.11	.63	.67	.84
Italy	.64	.10	.39	.50	.55
Japan	.52	.00	.04	.33	.95
Netherlands	.69	.33	.17	.83	.27
New Zealand	.56	.40	.54	.17	.15
Norway	.95	.95	.53	.83	.95
Sweden	.98	.98	1.00	.95	.70
Switzerland	.53	.34	.13	.67	.10
United Kingdom	.63	.61	.34	.50	.15
United States	.09	.00	.04	.05	.05

Figure 6 From Fuzzy Set Social Science Ragin 2000 p. 292

Let us now turn to the initial conditions. These are the transformed scores on the property types for the list of selected states. As we just said, having a generous welfare state is the outcome and the property distributions are the set of causal conditions. *f*QCA takes these property distributions and looks for alignments with the outcome property. If the condition score for a case is equal to or greater than the outcome, the outcome is taken to be a subset of the condition and hence necessary for the outcome property for this case. The overall 'necessity' score for the condition is the proportion of the scores where this inference holds. A filter is then applied to the calculated proportions. This is a "probabilistic test", as Ragin (p.295) calls it, which uses a threshold of 0.8 (set by Ragin) to which he allocates a significance level of p = 0.05. In other words, in variable analysis-speak, if the proportion is above 0.8, there is a 95% chance the result is not random and we can accept the proposition (null hypothesis) that the condition is necessary. As one of its boundary conditions, QCA insists that the identification of a set of countries with the condition creates its inverse, the set without the condition. The analysis tests both sets, meaning there are a possible 8 threshold proportions having a significance of p = 0.05 and above. Another way to describe this is to imagine the total set of countries mapped onto a 0:1 Cartesian space defined on the x axis by the relevant condition (say, strong unions) and on the y by the outcome (welfare generosity). What fQCA is looking for are those outcome values above 0.8 located in the lower right-hand triangle of the space where the x value equals or exceeds the y. Ragin talks of distributions like this as "corners".

None of the conditions satisfy this test and so Ragin concludes there are no necessary conditions. He explains this result by pointing to the lack of diversity in the collection of countries selected. They are all 'Western' AIDCs (a point we made right at the beginning). One feature usually associated with being such an AIDC is some level of welfare provision.

The method for identifying sufficient conditions is similar but more convoluted. The conditions are tested singly and in combination to see if they are subsets of the outcome. This is done by checking whether their scores on the property are equal to or less than the outcome (this time we are looking in the upper left triangle of the property space). If the property is a subset of the outcome, it shows it is one of possible ways the outcome might be generated. Once again both positive and negative conditions are tested. In the social welfare example, the combinatorics give us 3^4-1 (80) tests to run. The result is a list of approximately 40 instances of singleton properties or combinations which satisfy the 'test'. To simplify the analysis, *f*QCA looks for common factors in the combinations. If those common factors are themselves in the set, then their combinations are deleted because they are held to be logically redundant. The principle is: if condition A is sufficient, the combination of A with any other condition will also be sufficient. This is the application of sociology avoidance. Using the 'rule' allows Ragin to avoid the complications of providing sociological reasons why particular combination should be deleted. Truncating the properties in this way leaves the following as sufficient conditions for generous welfare provision: strong left parties; the intersection of strong unions and sociocultural homogeneity; the intersection of {the intersection of strong unions and corporatist industrial system} and absence of sociocultural homogeneity.

With all this in place, the completed analysis consists in identifying and describing patterns across the cases (which are close to which in regard of what conditions) and providing a

Table 10.7
Fuzzy Membership of Countries in Causal Expressions Passing Sufficiency

Country	Strong Left Parties	Strong Unions• Corporatist• ~Homogeneous	Strong Unions• Homogeneous	Corporatist• Homogeneous	Maximum of Causal Expression	Generous Welfare States
Australia	.25	.17	.25	.17	.25	.26
Austria	.70	.33	.64	.67	.70	.72
Belgium	.54	.71	.29	.29	.71	.79
Canada	.00	.05	.06	.05	.06	.26
Denmark	.85	.14	.81	.83	.85	.86
Finland	.56	.28	.72	.72	.72	.76
France	.12	.10	.10	.31	.31	.57
Germany	.43	.20	.20	.30	.43	.68
Ireland	.11	.16	.63	.67	.67	.67
Italy	.10	.39	.39	.50	.50	.64
Japan	.00	.04	.04	.33	.33	.52
Netherlands	.33	.17	.17	.27	.33	.69
New Zealand	.40	.17	.15	.15	.40	.56
Norway	.95	.05	.53	.83	.95	.95
Sweden	.98	.30	.70	.70	.98	.98

Figure 7 Ragin 2000 p. 298

socio-political historical narrative for why this might be the case. But this only takes us back to where the analysis started, namely the presumed similarities between clusters of the countries chosen; the Scandinavian countries; EU countries and Australia, Canada and USA. Japan and Switzerland remain the outliers.

DISCUSSION

We have provided far more detail on *f*QCA than the other examples we have looked at with good reason. Compared to them, and certainly compared to our idealised toy example, problem set-up and delineation of boundary and initial conditions involve a great deal more numerical 'processing' and 'transformation'. This is not simply because of the complications of the cases and the necessity of calibration. It is a direct result of an inability to ensure a clean severing at the problem's boundary.[8] The confounding of definitions, the informal basis for determining membership values, the imprecise characterisation of the objects in the property spaces as well as the odd ways the vectors of 'causal' conditions are computed, all contribute to a weakening of the account's plausibility structure. The system laws *f*QCA uses are contained in the arithmetic of fuzzy logic used to create and analyse the final lists of conditions given in the table above. We have said nothing about how well Ragin's usage conforms to the usual requirements of that mode of analysis. Since fuzzy logic is a well-defined domain in Mathematics, we assume the difficulties we are presented with are produced by the way it is used not the mathematics itself.

Section 3. Conclusion

The objective of this discussion was to offer a method for reading sociological reports which would allow an understanding of the extent to which the structure of the sociological rendering preserves the structure of the social objects it is attempting to describe. Our shorthand for this condition was a report's "plausibility". The method we suggested focuses on three key elements: the posing of the analytic problem, the procedures for effacement of non-relevant detail and the analytic procedures deployed. We have applied the method to a small number of different types of mathematical model building. Using the method to guide our reading, we identified several potential difficulties which those who apply mathematical methods within sociological analysis will have to overcome if their reports are to be structure preserving. Most, but not all, are consequences of the dependence on arithmetic of the mathematics being applied and its grounding in $\mathbb{R}^N$, the real number system. These problems lie both in the degree to which the problems can be 'well-posed' and the extent to

[8] This may sound harsh, but it is not as harsh as some well-informed commentators have been about Ragin's method. See [Lieberson 2004].

which a 'clean cut' can be achieved between the 'world-for-analysis' and 'the social world-for-investigation'.

The advantage of the heuristic we offer is that it focuses attention on key aspects of a report's construction. These aspects underpin its plausibility as a sociological account of the social phenomena being examined. As such, we think it might be a helpful addition to a student's analytic armoury when first encountering sociological reports for themselves (other than when summarised in texts, say). That is the spirit in which we recommend it. If it has further values (as we think it may well have), hopefully they will be demonstrated in the companion pieces included within this Part and later in this collection.

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